On dual processes for additive and monotone interacting particle systems and applications

Anja Sturm

University of Göttingen
Institute of Mathematical Stochastics

Joint work with Jan Swart (UTIA Prague)

Workshop Spatial models in population genetics
University of Bath
September 6-8, 2017
Outline

1. Duality concepts
2. A general construction of pathwise duals
3. Pathwise duality for monotone and additive processes
Outline

1. Duality concepts
2. A general construction of pathwise duals
3. Pathwise duality for monotone and additive processes
The classical duality concept

Let $X$ and $Y$ be two stochastic processes on some state spaces $S$ and $S'$.

$X$ and $Y$ are dual to each other with duality function $\psi$ if for $x \in S$ and $y \in S'$

$$\mathbb{E}^X[\psi(X_t, y)] = \mathbb{E}^Y[\psi(x, Y_t)], \quad t \geq 0.$$ 

(Roughly) equivalent:

- $G\psi = H\psi$ for $G$ and $H$ the generators of $X$ and $Y$
- $s \mapsto \mathbb{E}[\psi(X_s, Y_{t-s})]$ is constant on $[0, t]$ with $t \geq 0$ when $X$ and $Y$ are independent.

Remark: **Sub/superduality** if equality is replaced by inequality.
Generalization of the concept: Pathwise duality

$Y$ is a \textbf{(strong) pathwise dual} to $X$ with duality function $\psi$
if $X$ and $Y$ can be coupled such that

$$s \mapsto \psi(X_s, Y_{t-s})$$

is \textbf{almost surely constant} on $[0, t]$ with $t \geq 0$, and $X_{s-}$ is independent of $Y_{t-s}$, $s \in [0, t]$.

**Terminology, overview:** Jansen and Kurt '14
More literature and examples later.

In \textbf{particle system/population genetics} context dual running backwards into the past as ancestral/genealogical process.
Outline

1. Duality concepts

2. A general construction of pathwise duals

3. Pathwise duality for monotone and additive processes
Let $X$ be a continuous-time Markov chain with (finite) state space $S$ and generator $G$. Then $G$ can be written in the form of a **random mapping representation**:

Let $G \subset \mathcal{F}(S, S) := \{m : S \to S\}$ and let $(r_m)_{m \in G}$ be nonnegative constants.

$$Gf(x) = \sum_{m \in G} r_m (f(m(x)) - f(x)) \quad , x \in S.$$  

**Note:** This kind of representation is not unique.

The random mapping representation can be used for a Poissonian construction of the Markov process ($\to$ stochastic flow).
Poissonian construction of Markov processes

Let $\Delta$ be a Poisson point subset of $G \times \mathbb{R}$ with local intensity $r_m dt$. For $s \leq u$, set $\Delta_{s,u} := \Delta \cap (G \times (s, u])$.

Define random maps $X_{s,t} : S \to S$ ($s \leq t$) by

$$X_{s,t}(x) := m_n \circ \cdots \circ m_1(x)$$

when

$$\Delta_{s,t} := \{(m_1, t_1), \ldots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n.$$

Note that $X_{t,u} \circ X_{s,t} = X_{s,u}$ for all $s \leq t \leq u$.

Poisson construction of Markov processes
Let $X_0$ be an $S$-valued r.v., independent of $\Delta$. Setting for $s \in \mathbb{R}$,

$$X_t := X_{s,s+t}(X_0), \quad t \geq 0$$

defines a Markov process $X = (X_t)_{t \geq 0}$ with generator $G$. 
Pathwise duality from the Poissonian construction

Let $X$ and $Y$ be continuous-time Markov chains with (finite) state spaces $S$ and $S'$ and generators

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m(f(m(x)) - f(x)),$$

$$Hf(y) = \sum_{m \in \mathcal{G}} r_m(f(\hat{m}(y)) - f(y)).$$

Proposition (Pathwise duality)

Let $\psi : S \times S' \to \mathbb{R}$ be a function such that

\[(\ast) \quad \psi(m(x), y) = \psi(x, \hat{m}(y)) \quad x \in S, \ y \in S', \ m \in \mathcal{G}.\]

Then, $X$ and $Y$ are pathwise dual.

Proof: Use the Poissonian construction.
Construction of a pathwise dual

Goal:
- Construct in a general setting \( \hat{m} \) and \( \psi \) such that (*) holds:

\[
\psi(m(x), y) = \psi(x, \hat{m}(y)).
\]

General possibility Let \( S' = \mathcal{P}(S) \), the set of all subsets of \( S \), and

\[
\hat{m}(A) = m^{-1}(A) := \{ x \in S : m(x) \in A \}, \quad A \in \mathcal{P}(S).
\]

Then equality holds in (*) with respect to the duality function

\[
\psi(x, A) := 1_{\{x \in A\}}, \quad x \in S, A \in \mathcal{P}(S).
\]
General duality function

\[ \psi(x, A) := 1_{\{x \in A\}}, \quad x \in S, A \in \mathcal{P}(S). \]

"The dual with state space \( \mathcal{P}(S) \) tracks the set of configurations that a particular (set of) configuration(s) may have emerged from."

This dual may be too unwieldy. ⇒ Restrict the setting!

Find subspaces of \( \mathcal{P}(S) \) that are invariant under the inverse image maps \( m^{-1} \) for all \( m \in \mathcal{G} \).

Focus:
- Monotone and additive functions \( m \) on partially ordered sets.
Outline

1. Duality concepts
2. A general construction of pathwise duals
3. Pathwise duality for monotone and additive processes
Let \((S, \leq)\) be a (finite) partially ordered set.

- For \(A \subset S\) define \(A^\downarrow := \{x \in S : x \leq y \text{ for some } y \in A\}\).
- \(\mathcal{P}_{\text{dec}}(S)\) are the **decreasing** sets \(A\) with \(A^\downarrow \subset A\).
- \(\mathcal{P}_{\text{dec}}(S)\) is a **principal ideal** if it consists of \(A\) with

\[
A = \{z\}^\downarrow \text{ for some } z \in S.
\]

Define analogously \(A^\uparrow\),

**increasing** sets \(\mathcal{P}_{\text{inc}}(S)\) and **principle filters** \(\mathcal{P}_{\text{inc}}(S)\).
Little excursion: Partially ordered sets

- In a join-semilattice $\mathcal{P}_{\text{inc}}(S)$ is closed under finite intersections and the \textbf{supremum} is well defined via
  \[ \{x \lor y\}^\uparrow := \{x\}^\uparrow \cap \{y\}^\uparrow \]

- $x \lor y$ is the minimal element such that
  \[ x \leq x \lor y \quad \text{and} \quad y \leq x \lor y. \]

- For $S$ finite or bounded join-semilattice we have
  \[ \emptyset \neq A \subset \mathcal{P}_{\text{dec}}(S) \iff A \subset \mathcal{P}_{\text{dec}}(S) \text{ and } x, y \in A \text{ implies } x \lor y \in A. \]

**Example:**

In the context of interacting particle systems choose for example

- $S = \mathcal{P}(\Lambda)(\cong \{0, 1\}^\Lambda)$ with partial order $\subset$.

- Here, $\lor$ corresponds to $\cup$. 
Little excursion: Monotone and additive functions

- A function $m$ is **monotone** if
  \[ x \leq y \implies m(x) \leq m(y), \quad x, y \in S. \]

- A function $m$ is **additive** on a join-semilattice with minimal element 0 if
  \[ m(x \lor y) = m(x) \lor m(y), \quad x, y \in S \]
as well as $m(0) = 0$.

**Remark:**
- Additive functions are monotone.
Invariant subspaces for monotone and additive functions

Proposition (Monotone functions)
Equivalent:
- \( m \) is monotone.
- \( m^{-1} \) maps \( \mathcal{P}_{\text{dec}}(S) \) into itself (invariant subspace!).
- \( m^{-1} \) maps \( \mathcal{P}_{\text{inc}}(S) \) into itself (invariant subspace!).

Proposition (Additive functions)
Equivalent (on a finite join-semilattice with minimal element):
- \( m \) is additive.
- \( m^{-1} \) maps \( \mathcal{P}_{\text{!dec}}(S) \) into itself (invariant subspace!).

- \( m^{-1}(A) \in \mathcal{P}_{\text{dec}}(S) \) for \( A \in \mathcal{P}_{\text{!dec}}(S) \) (additive functions monotone)
- \( x, y \in m^{-1}(A) \Rightarrow x \lor y \in m^{-1}(A) \)
  since \( m(x \lor y) = m(x) \lor m(y) \) and \( m(x) \lor m(y) \in A \).
Monotonically and additively representable processes

If a Markov process $X$ has random mapping representation

$$Gf(x) = \sum_{m \in G} r_m(f(m(x)) - f(x)) \quad , x \in S$$

where

- $G$ contains only monotone functions then we call $X$ **monotonically representable**.
- $G$ contains only additive functions then we call $X$ **additively representable**.
Pathwise duality for additively representable processes

$S'$ is a *dual* of $S$ if there is a bijection $S \ni x \mapsto x' \in S'$ ($x'' = x$) with

$$x \leq y \iff x' \geq y'.$$

**Examples:**

- 1 $S' := S$ equipped with the reversed order and $x' = x$.
- 2 For $S \subset \mathcal{P}(\Lambda)$ equipped with $\subseteq$ take for $x' := \Lambda \setminus x = x^C$, the complement of $x$, and $S' := \{x' : x \in S\}$.

Now consider for $x \in S$, $y \in S'$

$$\psi(x, y) = 1_{\{x \leq y\}} = 1_{\{y \leq x'\}}$$

- 1 $\psi(x, y) = 1_{\{x \leq y'\}} = 1_{\{x \leq y\}}$
  Siegmund’s duality on a totally ordered space $S$
- 2 $\psi(x, y) = 1_{\{x \subset \Lambda \setminus y\}} = 1_{\{x \cap y = \emptyset\}}$
  Additive interacting particle systems
Pathwise duality for additively representable processes

Lemma (Duals to additive maps)

For additive \( m : S \to S \) there exists (a unique) \( m' : S' \to S' \) with

\[
(*) \quad 1_{\{m(x) \leq y'\}} = 1_{\{x \leq (m'(y))'\}}, \quad x \in S, y \in S'.
\]

Proof

- By additivity \( m^{-1} \) maps sets of the form
  \[
  A = \{y'\}^\uparrow = \{x \in S : x \leq y'\}, \quad y \in S'
  \]
  into sets of this form.

- Thus, there exists an element \( z \in S \) such that
  \[
  m^{-1}(\{x \in S : x \leq y'\}) = \{x \in S : x \leq z\}
  \]

Set \( m'(y) = z', y \in S' \)

\[
\Leftrightarrow m(x) \leq y' \quad \text{if and only if} \quad x \leq (m'(y))'
\]
Pathwise duality for additively representable processes

**Theorem (Additive systems duality)**

Let $S$ be a finite lattice and let $X$ be a Markov process in $S$ whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m \in G} r_m(f(m(x)) - f(x)), \quad x \in S,$$

where all maps $m \in G$ are additive (additively representable). Then the Markov process $Y$ in $S'$ with generator

$$Hf(y) := \sum_{m \in G} r_m(f(m'(y)) - f(y)), \quad y \in S'$$

is pathwise dual to $X$ with respect to the duality function

$$\psi(x, y) = 1\{x \leq y\}', \quad x \in S, y \in S'.$
Percolation structure for additively representable processes

Equip $S := \mathcal{P}(\Lambda)$ with $\subset$ and let $m$ be an additive map $S \to S$. Define $M \subset \Lambda \times \Lambda$ via

$$m(x) = \{ j \in \Lambda : (i, j) \in M \text{ for some } i \in x \} \quad x \in S.$$ 

Vice versa, any such $M \subset \Lambda \times \Lambda$ corresponds to an additive map $m$. 
Percolation structure for additively representable processes

Let $S' = S$ and $x' = x^C$. Then we have an additive $m' : S \to S$ dual to $m$ with the duality function

$$\psi(x, y) = 1_{\{x \subset \Lambda \setminus y\}} = 1_{\{x \cap y = \emptyset\}}, \quad x, y \in S.$$ 

The $M' \subset \Lambda \times \Lambda$ corresponding to $m'$ is given by

$$M' = \{(j, i) : (i, j) \in M\}.$$
Percolation structure for additively representable processes

Percolation representation

Plot space-time $\Lambda \times \mathbb{R}$ with time upwards.
At rate $r_m$ we consider the $M$ associated to $m$ and
  - draw an arrow from $(i, t)$ to $(j, t)$ ($i \neq j$) whenever $(i, j) \in M$
  - place a “blocking symbol” at $(i, t)$ whenever $(i, i) \notin M$

”Open paths” $\rightsquigarrow$ travel upwards along arrows and avoid blocking symbols. Then

$$X_{s,u}(x) = \{ j \in \Lambda : (i, s) \rightsquigarrow (j, u) \text{ for some } i \in x \},$$

and the dual process is obtained via open paths using the reversed arrows (in reversed time).
Voter model

\[ S = \{0, 1\}^\Lambda \cong \mathcal{P}(\Lambda). \]
Percolation structure for additively representable processes

Extensions
The above percolation structure statements also apply if

- Λ is a partially ordered set and $S = \mathcal{P}_{\text{dec}}(\Lambda)$.
- $S$ is a distributive lattice with

$$x \land (y \lor z) = (x \land y) \lor (x \land z) \quad x, y, z \in S.$$ 

One can show that $S \cong \mathcal{P}_{\text{dec}}(\Lambda)$ for a partially ordered set $\Lambda$ by Birkhoff’s representation theorem.

In this case for $i, j, i', j' \in \Lambda$

(i) $(i, j) \in M$ and $i \leq i'$ implies $(i', j) \in M$, 
(ii) $(i, j) \in M$ and $j \geq j'$ implies $(i, j') \in M$. 

Percolation structure for additively representable processes

Two stage contact process (Krone '99)

$S = \{0, 1, 2\}^\Lambda$ "1" younger individual "2" older individual. Older individuals give birth to younger individuals who "grow up" and possibly die at a higher rate than older individuals.

$S \cong \mathcal{P}_{\text{dec}}(\Lambda \times \{0, 1\})$. 
Pathwise duality for monotonically representable processes

Now consider the duality function

$$\phi(x, B) := 1 \{ x \leq y' \text{ for some } y \in B \}, \quad x \in S, \ B \in \mathcal{P}(S').$$

Lemma (Duals to monotone maps)

For monotone $m : S \to S$ there exist $m^* : \mathcal{P}(S') \to \mathcal{P}(S')$ with

$$(*) \quad 1 \{ m(x) \leq y' \text{ for some } y \in B \} = 1 \{ x \leq y' \text{ for some } y \in m^*(B) \}.$$  

Proof idea

- By monotonicity $m^{-1}$ maps decreasing sets of the form

  $$A = \{ B' \} = \{ x \in S : x \leq y' \text{ for some } y \in B \}, \quad B \in \mathcal{P}(S')$$

  into sets of this form.

- Construct appropriate $m^* : m^*(B)' := \bigcup_{x \in B} (m^{-1}(\{ x' \}))_{\text{max}}$
Theorem (Monotone systems duality)

Let $S$ be a finite partially ordered set and let $X$ be a Markov process in $S$ whose generator has a random mapping representation of the form

$$Gf(x) = \sum_{m \in G} r_m(f(m(x)) - f(x)) \quad x \in S,$$

where all maps $m \in G$ are monotone (monotonically rep.). Then the $\mathcal{P}(S')$-valued Markov process $Y^*$ with generator

$$H^*_f(B) = \sum_{m \in G} r_m(f(m^*(B)) - f(B)), \quad B \in \mathcal{P}(S')$$

is pathwise dual to $X$ with respect to the duality function $\phi$. 
State space

- $\Lambda = (V, E)$ be a countable, connected, vertex transitive (degree $D$), locally finite graph with vertex set $V$ and set of (undirected) edges $E$
- $S = \mathcal{P}(\Lambda) \cong \{0, 1\}^\Lambda$

Examples:

- $\Lambda = \mathbb{Z}^d$ with nearest-neighbor edges ($D = 2d$)
- $\Lambda = K_N$ complete graph ($D = N - 1$)
- $\Lambda = \mathbb{T}_d$ a regular tree ($D = d + 1$)
Pathwise duality for cooperative branching coalescent

Continuous-time Markov process $X = (X_t)_{t \geq 0}$ on $\{0, 1\}^\Lambda \cong \mathcal{P}(\Lambda)$

- **Pairs of particles produce a new particle:** $110 \rightarrow 111$
  
  map $\text{coop}_{ijk}$ for $\langle i, j \rangle, \langle j, k \rangle \in E$ at rate $\beta \frac{1}{D(D-1)}$

  for particles at sites $i$ and $j$ producing a particle at site $k$

- **Symmetric random walk with coalescence:** $10, 11 \rightarrow 01$
  
  map $\text{rw}_{ij}$ for $\langle i, j \rangle \in E$ at rate $\gamma \frac{1}{D}$

  particle moving from $i$ to $j$ merging with any particle present

- **Spontaneous death of particles:** $1 \rightarrow 0$
  
  map $\text{death}_i$ at rate $\delta$

  particle at site $i$ disappears

**Remark:** One may also include voter $\text{vot}_{ij} : 01 \rightarrow 11, 10 \rightarrow 00$

and exclusion dynamics $\text{exc}_{ij} : 10 \rightarrow 01, 01 \rightarrow 10$
Pathwise duality for cooperative branching coalescent

Examples considered:

- $\Lambda = \mathbb{Z}$ without spontaneous death:
  Sturm, Swart ’15

- $\Lambda = K_N$ complete graph without random walk
  (also $\Lambda = \mathbb{T}_d, \mathbb{Z}^d$):
  Mach, Sturm, Swart, in progress ’17
Pathwise duality for cooperative branching coalescent

All maps $m$ are monotone, all but cooperative branching are additive. Let $S' = S$ and $x' = x^C$. Then the duality function is

$$
\phi(x, B) = 1_{\{x \subset y^C \text{ for some } y \in B\}} = 1_{\{x \cap y = \emptyset \text{ for some } y \in B\}}
$$

for $x \in S, B \in \mathcal{P}(S)$.

For the additive functions $m$ there are dual functions $m'$ with

$$
m(x) \cap y = \emptyset \iff x \cap m'(y) = \emptyset
$$

and we set $m^*(B) = \{m'(x) : x \in B\}$. We have

$$
\text{rw}'_{ij} = \text{vot}_{ij}, \quad \text{death}'_i = \text{death}_i, \quad \text{vot}'_{ij} = \text{rw}_{ij}, \quad \text{exc}'_{ij} = \text{exc}_{ij}
$$
Pathwise duality for cooperative branching coalescent

For the cooperative branching map we have

$$\text{coop}_{ijk}^*(B) = b_{ijk}^{(1)}(B) \cup b_{ijk}^{(2)}(B)$$

with the definition (restricted to sites $ijk$)

$$b^{(1)} : 001 \to 011, \quad b^{(2)} : 001 \to 101$$

since

$$\left(\text{coop}^{-1}(\{x\}^\uparrow)\right)_{\max} = \begin{cases} 
\{100, 010\} & \text{if } x = 110, \\
\{x\} & \text{otherwise.}
\end{cases}$$

and $x' := x^C$. 
Sturm, Swart '15
\( \Lambda = \mathbb{Z} \) without spontaneous death

- **Application of a version of this dual:**
  Decay rates of the survival probability and the density in the subcritical regime is order \( t^{-1/2} \)

- **Additional results regarding phase transitions**

  \[
  \beta_{\text{surv}} := \inf\{\beta > 0 : \text{the process survives}\}, \\
  \beta_{\text{upp}} := \inf\{\beta > 0 : \text{the upper invariant law is nontrivial}\}.
  \]

  We have \( 1 \leq \beta_{\text{surv}}, \beta_{\text{upp}} < \infty \).
Pathwise duality for cooperative branching coalescent

Let $S' = S$ with reversed order and consider the duality function

$$\tilde{\phi}(x, B) = 1 \{ x \geq y \text{ for some } y \in B \}, \quad x \in \{0, 1\}^\Lambda, B \in \mathcal{P}(S).$$

By considering $(m^{-1}(\{x\}^\uparrow))_{\text{min}}$ obtain the dual maps

- **Double branching map** $\text{coop}^*_{ijk}(B) = B \cup \text{dbran}_{ijk}(B)$
  with the map $\text{dbran}_{ijk} : 001, 011, 101, 111 \rightarrow 110$

- **Random walk map** $\text{rw}^*_{ij}(B) := \{ y \in B : y(i) = 0 \} \cup \text{e}_{ij}(B)$
  with the map $\text{e}_{ij} : 01 \rightarrow 10$

- **Death map** $\text{death}^*_i(B) := \{ y \in B : y(i) = 0 \}$. 
Pathwise duality for cooperative branching coalescent

Mach, Sturm, Swart ’17+
Model with cooperative branching and spontaneous death.

- Application of this dual to characterize the behavior of the process and its dual on $K_N$ for $N \to \infty$ (mean field model).
- $\beta_{upp} < \beta_{surv}$ on $\mathbb{T}_d$ with $d \geq 9$.
- $\beta_{upp} \leq \beta_{surv}$ on $\mathbb{Z}_d$ (conjecture $\beta_{upp} = \beta_{surv}$).
Conclusion:

- General framework for obtaining duals, in particular for (monotone, additive) spatial interacting particle systems.
- Some duals may be interpreted as potential ancestors/genealogies.