Ancestral lineages in locally regulated populations

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based on joint work, in part in progress, with
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Outline

1 Introduction

2 Discrete time contact process and oriented percolation
   - Ancestral lines, RWDRE and a CLT
   - Central proof ingredient: Regeneration construction

3 Logistic branching random walks and ‘relatives’
   - Long-time behaviour: Coupling and convergence
   - Ancestral lineages in the spatial logistic model
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   - Ancestral lineages in the spatial logistic model
Stepping stone model (Kimura, 1953) (here, in discrete time)

Colonies of fixed size $N$ are arranged in a geographical space, say $\mathbb{Z}^d$

$$\vdots \leftarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \vdots$$

($d = 1$ in this picture)
Stepping stone model (Kimura, 1953) (here, in discrete time)

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$\cdots$ $\rightarrow$ $\cdots$

$(d = 1$ in this picture)$

For each child: Assign a random parent in same colony with probability $1 - \nu$, in a neighbouring colony with probability $\nu$
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More generally, at for each individual in colony $x$, with probability $p(x, y) = p(y - x)$ assign a random parent in previous generation from colony $y$
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“Trivial” demographic structure, but paradigm model for evolution of type distribution in space
Stepping stone model: Ancestral lines
The stepping stone model

Fixed local population size $N$ in each patch (arranged on $\mathbb{Z}^d$), patches connected by (random walk-type) migration

Pros:  
+ Stable population, no local extinction, nor unbounded growth
+ Ancestral lineages are (delayed) coalescing random walks (in particular, well defined),
   this makes detailed analysis feasible, yields via duality: long-time behaviour of (neutral) type distribution
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**Cons:**

− An ‘ad hoc’ simplification, effects of local size fluctuations not explicitly modelled
− $N$ is an ‘effective’ parameter, relation to ‘real’ population dynamics is unclear
− Grid not so realistic for most populations
Remark: A problem with branching random walk

(Critical) branching random walks, where particles move and produce offspring independently, explicitly model fluctuations in local population size, but do not allow stable populations in $d \leq 2$:

(Felsenstein’s “pain in the torus” 1975; Kallenberg 1977)
Remark: A problem with branching random walk, 2

One could try to slow down the branching:

**Self-catalytic critical branching random walks** (in continuous time)

\[ a = (a_x)_{x \in \mathbb{Z}^d} \ldots \text{a probability kernel} \]

\[ b : \mathbb{N}_0 \to [0, \infty) \ldots \text{branching rate function, } b(0) = 0 \]
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- Particles perform independent continuous-time random walks on \( \mathbb{Z}^d \)
  - with jump rate 1 and jump increments drawn from \( a \)
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- When currently \( k \) particles at \( x \), at rate \( b(k) \) one of them performs critical (binary) branching, i.e.
  - disappears or doubles, each with prob. \( \frac{1}{2} \)
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  note: each particle branches then at rate \( b(k)/k \),
  
  \[ b(k) = c \cdot k \text{ corresponds to independent branching rw} \]
Even \( b(k) \ll k \) cannot prevent clustering

Self-catalytic branching rw, \( b(k) = k^{1/10} \) in \( d = 1 \) (on \( \mathbb{Z}/(500\mathbb{Z}) \))
Even $b(k) \ll k$ cannot prevent clustering

\[ \eta_x(t) \ldots \text{number of particles at position } x \in \mathbb{Z}^d \text{ at time } t \geq 0, \]

assume $\sup_{x \in \mathbb{Z}^d} \mathbb{E}[\eta_x(0)] < \infty$
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\[ \eta_x(t) \] ... number of particles at position \( x \in \mathbb{Z}^d \) at time \( t \geq 0 \),

assume \( \sup_{x \in \mathbb{Z}^d} \mathbb{E}[\eta_x(0)] < \infty \)

**Theorem** (B. & Sun, 2017).

If \( b(1) > 0 \)
and motion with the symmetrised kernel \( \hat{a}_x = (a_x + a_{-x})/2 \) is recurrent (e.g. finite variance and \( d \leq 2 \)),

\[
\lim_{t \to \infty} \mathbb{P}(\eta_x(t) = 0) = 1 \quad \text{for all } x \in \mathbb{Z}^d.
\]
A simulation, \( b(k) = 1_{\{k=1\}} \)
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The discrete time contact process

\[ \eta_n(x), \ n \in \mathbb{Z}_+, \ x \in \mathbb{Z}^d, \ \text{values in} \ \{0, 1\}. \]

Site \( x \) is generation \( n \) is “inhabited” (or: “infected”) if \( \eta_n(x) = 1 \).
The discrete time contact process

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Dynamics: \( U = \{ y \in \mathbb{Z}^d : \|y\|_\infty \leq 1 \} \subset \mathbb{Z}^d \) finite, symmetric, \( p \in (0, 1) \).

Given \( \eta_n \), independently for \( x \in \mathbb{Z}^d \),

\[
\eta_{n+1}(x) = \begin{cases} 
1 \text{ w. prob. } p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U) \\
0 \text{ w. prob. } 1 - p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U)
\end{cases}
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\]

**Interpretation:**

In generation \( n + 1 \), each site \( x \) is (independently) inhabitable with probability \( p \).

If \( \eta_n(y) = 1 \) for some \( y \in x + U \), the particle at \( y \) in generation \( n \) places an offspring at \( x \).

If several \( y \) are eligible, one is chosen at random.
The discrete time contact process …

… viewed as a locally regulated population model

Neighbours compete for inhabitable sites, so individuals in sparsely populated regions have on average higher reproductive success.
The discrete time contact process ... 

... viewed as a locally regulated population model

Neighbours compete for inhabitable sites, so individuals in sparsely populated regions have on average higher reproductive success.

This is particularly evident in the multitype version, where particles carry a type, e.g. from (0, 1), and offspring inherit parent’s type.

\[ n + 1 \]

\[ n \]

expected no. of red offspring:

\[ 3p > 1 \]

expected no. of red offspring:

\[ 3\frac{1}{3}p = p < 1 \]
Alternative view: Directed (site) percolation

\( \omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}, \text{i.i.d. Bernoulli}(p) \)

Interpretation: \( \omega(x, n) = 1 \) : site \((x, n)\) is inhabitable/open, otherwise not inhabitable/closed
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Open paths:

\( m < n, \ x, y \in \mathbb{Z}^d : (x, m) \to_\omega (y, n) \) if there exist \( x = x_0, x_1, \ldots, x_{n-m} = y \)
such that \( ||x_i - x_{i-1}||_\infty \leq 1 \) and \( \omega(x_i, m + i) = 1 \) for \( i = 1, \ldots, n - m \)
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\( C_0 := \{(y, n) : y \in \mathbb{Z}^d, \ n \geq 0, \ (0,0) \rightarrow_{\omega} (y, n)\} \) is the (directed) cluster of the origin
There exists $p_c \in (0, 1)$ such that

$$\mathbb{P}(|C_0| = \infty) > 0 \quad \text{iff} \quad p > p_c.$$
There exists $p_c \in (0, 1)$ such that

$$\mathbb{P}(\vert C_0 \vert = \infty) > 0 \text{ iff } p > p_c.$$ 

If $p > p_c$, $\mathbb{P}(C_0 \text{ reaches height } n \mid \vert C_0 \vert < \infty) \leq C e^{-cn}$ for some $c, C \in (0, \infty)$. 
Stationary contact process and directed percolation

Assume \( p > p_c \) (from now on).

Start with \( \eta_{-m}(y) \equiv 1 \) at time \( -m < 0 \), then \( (n > -m) \)

\[
\eta_n(x) = 1 \iff \exists y \in \mathbb{Z}^d : (y, -m) \rightarrow_{\omega} (x, n).
\]

\[\text{time } n\]

\[\text{time } -m\]
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$$\eta_n(x) = 1 \iff \exists y \in \mathbb{Z}^d : (y, -m) \rightarrow_\omega (x, n).$$

$m \rightarrow \infty$ yields $(\eta_n^{\text{stat}})_{n \in \mathbb{Z}}$, the stationary (discrete time) contact process

$$\eta_n^{\text{stat}}(x) = 1 \iff \mathbb{Z}^d \times \{-\infty\} \rightarrow_\omega (x, n)$$

(the law of $\eta_0^{\text{stat}}$ is the upper invariant measure, the unique non-trivial ergodic stationary distribution)
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An ancestral line in the stationary contact process

\((\eta_n^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z})\) stationary DCP, assume \(\eta_0^{\text{stat}}(0) = 1\).

Let \(X_n = \) position of the ancestor of the individual at the (space-time) origin \(n\) generations ago.
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\[\{y \in \mathbb{Z}^d : \|y - x\|_\infty \leq 1, \eta^{\text{stat}}_{-n-1}(y) = 1\} (\neq \emptyset)\.]
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\[ \{ y \in \mathbb{Z}^d : ||y - x||_\infty \leq 1, \eta_{-n-1}^{\text{stat}}(y) = 1 \} \ (\neq \emptyset). \]

To avoid lots of --signs later, put \(\xi_n(x) := \eta_{-n}^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z}\).

Note: \(\xi_n(x) = 1 \iff "(x, n) \rightarrow \mathbb{Z}^d \times \{+\infty\}"\)
Directed random walk on the supercritical oriented cluster

\( \omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}, \text{ i.i.d. Bernoulli}(p), p > p_c \)

\( \xi(x, n) \ ( = \xi(x, n; \omega)) = 1 \text{ iff } "(x, n) \rightarrow_\omega \mathbb{Z}^d \times \{+\infty\}" \)

Put \( C := \{(y, m) : \xi(y, m) = 1\} \) (the “backbone” of the oriented cluster, i.e. “dangling ends” are removed),

\( U(x, n) := \{y : \|y - x\|_\infty \leq 1\} \times \{n + 1\} \)
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Let \( X_0 = 0 (\in \mathbb{Z}^d) \),

\[ \mathbb{P}(X_{n+1} = y \mid \xi, X_n = x, X_{n-1} = x_{n-1}, \ldots X_1 = x_1) = \frac{1(y \in U(x, n) \cap C)}{|U(x, n) \cap C|} \]
Directed random walk on the supercritical oriented cluster

\[ \omega(x, n), \, x \in \mathbb{Z}^d, \, n \in \mathbb{Z}, \text{ i.i.d. Bernoulli}(p), \, p > p_c \]
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It turns out: \((X_n)\) is similar to “ordinary” random walk on large scales.
Directed random walk on the supercritical oriented cluster

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It turns out: \((X_n)\) is similar to “ordinary” random walk on large scales.

(It is a random walk in dynamic random environment, but somewhat non-standard from RWDRE point of view.)
LLN and CLT for directed walk on the oriented cluster

**Theorem** (B., Černý, Depperschmidt, Gantert 2013). Let $B_0 := \{(0, 0) \in C\}$, $p > p_c$. 

$\mathbb{P}\left(\frac{1}{n} X_n \rightarrow 0 \mid B_0\right) = 1$ and $\mathbb{P}\left(\frac{1}{n} X_n \rightarrow 0 \mid \omega\right) = 1$ for $\mathbb{P}(\cdot \mid B_0)$-a.a. $\omega$, there exists $\sigma \in (0, \infty)$ s.th.

$$
\lim_{n \to \infty} \mathbb{E}\left[f\left(\frac{1}{\sigma \sqrt{n}} X_n\right) \mid \omega\right] = \mathbb{E}\left[f(Z)\right] \quad \text{for } \mathbb{P}(\cdot \mid B_0)-\text{a.a. } \omega
$$

for any continuous bounded $f : \mathbb{R}^d \to \mathbb{R}$, where $Z$ is $d$-dimensional standard normal

(in particular $\mathbb{E}\left[f\left(\frac{1}{\sigma \sqrt{n}} X_n\right) \mid B_0\right] \xrightarrow{n \to \infty} \mathbb{E}\left[f(Z)\right]$).

An invariance principle holds as well.
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Proof ingredient: A local construction of the walk

For \( x \in \mathbb{Z}^d \), \( n \in \mathbb{Z} \) let \( \tilde{\omega}(x, n) = (\tilde{\omega}(x, n)[1], \tilde{\omega}(x, n)[2], \ldots, \tilde{\omega}(x, n)[3^d]) \) an independent uniform permutation of \( U(x) = \{ y : \| y - x \|_\infty \leq 1 \} \).
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For a space-time point \((x, n)\) and \(k \in \mathbb{N}\) define a (directed) path \( \gamma_k^{(x, n)} \) of \(k\) steps that \emph{begin} on open sites, choosing directions according to \( \tilde{\omega} \):

- \( \gamma_k^{(x, n)}(0) = x \),
- if \( \gamma_k^{(x, n)}(j) = y \) then \( \gamma_k^{(x, n)}(j + 1) = z \), where \( z \) is the element of

\[
\{z' : \|z' - y\|_\infty \leq 1, (z', n + j + 1) \rightarrow_\omega \mathbb{Z}^d \times \{n + k - 1\}\}
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with the smallest index in \( \tilde{\omega}(y, n + j) \).
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For a space-time point \((x, n)\) and \( k \in \mathbb{N} \) define a (directed) path \( \gamma_{k}^{(x, n)} \) of \( k \) steps that begin on open sites, choosing directions according to \( \tilde{\omega} \):

1. \( \gamma_{k}^{(x, n)}(0) = x \),
2. if \( \gamma_{k}^{(x, n)}(j) = y \) then \( \gamma_{k}^{(x, n)}(j + 1) = z \), where \( z \) is the element of
   \[ \{ z' : \|z' - y\|_\infty \leq 1, (z', n + j + 1) \rightarrow_\omega \mathbb{Z}^d \times \{ n + k - 1 \} \} \]
   with the smallest index in \( \tilde{\omega}(y, n + j) \)
Local vs global construction of the walk

\[ \gamma_{(x,n)}(k) = \text{endpoint of the local } k\text{-step construction} \]

For \((x, n) \in \mathcal{C}\),

\[ \gamma_{(x,n)}(j) := \lim_{k \to \infty} \gamma_{(x,n)}(k) \text{ exists } \forall j \]

and \(\gamma_{(x,n)}(k) = \gamma_{(x,n)}(k)\) if \(\xi_{n+k}(\gamma_{(x,n)}(k)) = 1\).
Local vs global construction of the walk

\[ \gamma_{k}^{(x,n)}(k) = \text{endpoint of the local } k\text{-step construction} \]

\[ (x, n) \]

\[ \tilde{\omega}_{(x,n)}[1] \rightarrow \tilde{\omega}_{(x,n)}[2] \]

For \((x, n) \in \mathcal{C},\) \( \gamma_{\infty}^{(x,n)}(j) := \lim_{k \to \infty} \gamma_{k}^{(x,n)}(j) \) exists \( \forall j \)

and \( \gamma_{k}^{(x,n)}(k) = \gamma_{\infty}^{(x,n)}(k) \) if \( \xi_{n+k}(\gamma_{k}^{(x,n)}(k)) = 1. \)

On \( B_0 := \{(0,0) \in \mathcal{C}\}, \)

\[ X_k := \gamma_{\infty}^{(0,0)}(k), \ k = 0,1,2,\ldots \]

is (a version of) the directed random walk on \( \mathcal{C}, \)

and \( X_k = \gamma_{k}^{(0,0)}(k) \) if \( \xi(\gamma_{k}^{(0,0)}(k), k) = 1. \)
Regeneration times

\[ T_0 := 0, \ Y_0 := 0, \]
\[ T_1 := \min \{ k > 0 : \xi(\gamma_k^{(0,0)}(k), k) = 1 \}, \ Y_1 := \gamma_{T_1}^{(0,0)}(T_1) = X_{T_1}, \]
Regeneration times

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\[ T_1 := \min \{ k > 0 : \xi(\gamma_k^{(0,0)}(k), k) = 1 \}, \ Y_1 := \gamma_{T_1}^{(0,0)}(T_1) = X_{T_1}, \]
then \[ T_2 := T_1 + \min \{ k > 0 : \xi(\gamma_k^{(Y_1,T_1)}(k), T_1 + k) = 1 \}, \]
\[ Y_2 := \gamma_{T_2-T_1}^{(Y_1,T_1)}(T_2 - T_1) = X_{T_2}, \] etc.
\(( (Y_i - Y_{i-1}, T_i - T_{i-1}) )_{i \geq 1} \) are i.i.d. under \( P(\cdot | B_0) \), \( Y_1 \) is symmetrically distributed. There exist \( C, c \in (0, \infty) \), such that
\[
P(||Y_1|| > n | B_0), \ P(T_1 > n | B_0) \leq Ce^{-cn} \quad \text{for } n \in \mathbb{N}.
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Tail bounds use the fact that finite clusters are small, i.i.d. property follows from the fact that the local path construction uses disjoint time-slices.
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Randomised version of Kuczek’s (1989) construction, morally a discrete time analogue of Neuhauser (1992)
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Tail bounds use the fact that finite clusters are small, i.i.d. property follows from the fact that the local path construction uses disjoint time-slices.

Randomised version of Kuczek’s (1989) construction, morally a discrete time analogue of Neuhauser (1992)

A.s. CLT uses variance estimates obtained from joint regeneration for two independent copies \((X_n), (X'_n)\) on the same \(\xi\).
A “meta-theorem”

“Everything”\(^1\) that is true for the neutral multi-type voter model is also true for the neutral multi-type contact process (and presumably for more general locally regulated models).

\(^1\)with a suitable interpretation of “everything”
A “meta-theorem”

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Some “rigorous instances” (S. Steiber 2017)

- Start two walks on the cluster in \(x_1 \neq x_2\) which coalesce when meeting (condition on \(\xi(x_1, 0) = \xi(x_2, 0) = 1\)), let \(T_{\text{merge}} = \text{time until coalescence}\)
  \[ T_{\text{merge}} < \infty \text{ a.s. in } d = 1, 2, \mathbb{P}(T_{\text{merge}} = \infty) > 0 \text{ in } d \geq 3 \]
  (in particular, no neutral multi-type equilibria in \(d \leq 2\))

- In \(d = 1\), \(\mathbb{P}(T_{\text{merge}} \geq n) \asymp \frac{|x_1 - x_2|}{\sqrt{n}}\),
  start coalescing directed random walks on every site with \(\xi(x, n) = 1\)
  and re-scale this set of paths diffusively:
  Obtain Brownian web as scaling limit.

\(^1\)with a suitable interpretation of “everything”
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   - Central proof ingredient: Regeneration construction

3. Logistic branching random walks and ‘relatives’
   - Long-time behaviour: Coupling and convergence
   - Ancestral lineages in the spatial logistic model
Logistic branching random walks

Particles ‘live’ in \( \mathbb{Z}^d \) in discrete generations, 
\( \eta_n(x) = \# \text{ particles at } x \in \mathbb{Z}^d \text{ in generation } n. \)

Given \( \eta_n \),
each particle at \( x \) has Poisson\( \left( (m - \sum_z \lambda_{z-x} \eta_n(z))^+ \right) \) offspring,
\( m > 1, \lambda_z \geq 0, \lambda_0 > 0, \text{ symmetric, finite range}. \)

(Interpretation as local competition:
Ind. at \( z \) reduces average reproductive success of focal ind. at \( x \) by \( \lambda_{z-x} \))
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Children take an independent random walk step to $y$ with probability $p_{y-x}$, $p_{xy} = p_{y-x}$ symmetric, aperiodic finite range random walk kernel on $\mathbb{Z}^d$. 


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$$\eta_{n+1}(y) \sim \text{Poi}\left(\sum_x p_{y-x} \eta_n(x) \left(m - \sum_z \lambda_{z-x} \eta_n(z)\right)^+\right), \quad \text{independent}$$
Remarks

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- For \( \lambda \equiv 0 \), \((\eta_n)\) is a branching random walk.

- \((\eta_n)\) is a spatial population model with local density-dependent feedback:
  - Offspring distribution supercritical when there are few neighbours, subcritical when there are many neighbours

- System is in general not attractive.

- Conditioning\(^2\) on \( \eta_n(\cdot) \equiv N \) for some \( N \in \mathbb{N} \) (“effective local population size”) yields a discrete version of the stepping stone model

- Form of competition kernel and Poisson offspring law are convenient but could be more general.

\(^2\) and considering types and/or ancestral relationships
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Survival and complete convergence

**Theorem** (B. & Depperschmidt, 2007).

Assume $m \in (1, 3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

$(\eta_n)$ survives for all time globally and locally with positive probability for any non-trivial initial condition $\eta_0$. Given survival, $\eta_n$ converges in distribution to its unique non-trivial equilibrium.
**Survival and complete convergence**

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Proof uses

- corresponding deterministic system

\[
\zeta_{n+1}(y) = \sum_x p_{y-x} \zeta_n(x) \left( m - \sum_z \lambda_{z-x} \zeta_n(z) \right)^+
\]

has unique (and globally attracting) non-triv. fixed point

- strong coupling properties of $\eta$

- coarse-graining and comparison with directed percolation
Remarks

- Local extinction for $m \leq 1$ (domination by subcritical branching r.w.)
- Restriction to $m < 3$ in result “inherited” from logistic iteration $w_{n+1} = mw_n(1 - w_n)$
- Survival can be proved also for $m \in [3, 4)$ with analogous arguments, convergence not
Coupling: An essential proof ingredient

$m = 1.5$, $p = (1/3, 1/3, 1/3)$, $\lambda = (0.01, 0.02, 0.01)$

Starting from any two initial conditions $\eta_0$, $\eta'_0$, copies $(\eta_n)$, $(\eta'_n)$ can be coupled such that if both survive, $\eta_n(x) = \eta'_n(x)$ in a space-time cone.
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Dynamics of an ancestral line

Given stationary \((\eta_n^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d)\), cond. on \(\eta_0^{\text{stat}}(0) > 0\) (and “enrich” suitably to allow bookkeeping of genealogical relationships), sample an individual from space-time origin \((0, 0)\) (uniformly)
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Let \( X_n = \) position of her ancestor \( n \) generations ago:

Given \( \eta^{\text{stat}} \) and \( X_n = x \), \( X_{n+1} = y \) w. prob.

\[
p_{x-y} \eta^{\text{stat}}_{-n-1}(y) (m - \sum_z \lambda_{z-y} \eta^{\text{stat}}_{-n-1}(z))^+ \\
\sum_{y'} p_{x-y'} \eta^{\text{stat}}_{-n-1}(y') (m - \sum_z \lambda_{z-y'} \eta^{\text{stat}}_{-n-1}(z))^+
\]

(note: a Poisson vector conditioned on its total sum is multinomial)
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**Question:**

\((X_n)\) is a random walk in a – relatively complicated – random environment. Is it similar to an ordinary random walk when viewed over large enough space-time scales?
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**Note:** \((X_n)\) is close to ordinary rw in regions where relative variation of \(\eta_{-n-1}(x)\) is small.
Large scale behaviour of an ancestral line

\( X_n = \) position of ancestor \( n \) generations ago of an individual sampled today at origin in equilibrium

**Theorem** (LLN and (averaged) CLT, B., Černý, Depperschmidt 2016).

If \( m \in (1, 3), 0 < \lambda_0 \ll 1, \lambda_z \ll \lambda_0 \) for \( z \neq 0 \),

\[
\mathbb{P}\left( \frac{1}{n} X_n \to 0 \mid \eta_0(0) \neq 0 \right) = 1 \quad \text{and} \quad \mathbb{E}\left[ f\left( \frac{1}{\sqrt{n}} X_n \right) \mid \eta_0(0) \neq 0 \right] \xrightarrow{n \to \infty} \mathbb{E}[f(Z)]
\]

for \( f \in C_b(\mathbb{R}^d) \), where \( Z \) is a \( d \)-dimensional normal rv.
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The proof uses a *regeneration* construction
(and *coarse-graining* and *coupling*, in particular with directed percolation):

Regeneration times \(0 = T_0 < T_1 < T_2 < \cdots,\) express \(X_{T_k} = Y_1 + \cdots + Y_k\) with \(Y_i := X_{T_i} - X_{T_{i-1}}\) and \((Y_i, T_i - T_{i-1})_{i \geq 1} \text{ ‘almost i.i.d.’}\)
Spatial population models ($\eta_n$) and ancestral lineages ($X_k$): Abstract conditions

- **Local Markov structure**: $\eta_{n+1}(x)$ is a function of $\eta_n$ in a finite window around $x$ plus ‘local randomness’

Given $\eta$, $(X_k)_{k=0,1,...}$ is a Markov chain, $\mathbb{P}(X_{k+1} = \cdot | \eta, X_k = x)$ depends on $\eta_{-k}$, $\eta_{-k-1}$ in a finite window around $x$

[Note reversal of time between $\eta$ and $X$]
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- **Good configurations and coupling propagation for $\eta$ on coarse-grained scale $L_{\text{space}}\mathbb{Z}^d \times L_{\text{time}}\mathbb{Z}$**: With high probability,

  ‘good’ blocks make neighbours good in $L_{\text{time}}$ steps and $\eta$’s with two different good local initial conditions become locally identical after $L_{\text{time}}$ steps

On good $\eta$ blocks, the law of $X$ is ‘well behaved’: close to a non-disorded symmetric finite range reference walk

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- **Symmetry in distribution**
Idea for constructing regeneration times

Find time points along the path such that:

- a cone (with fixed suitable base diameter and slope) centred at the current space-time position of the walk covers the path and everything it has explored so far (since the last regeneration)
- configuration $\eta^{\text{stat}}$ at the base of the cone is “good”
- “strong” coupling for $\eta^{\text{stat}}$ occurs inside the cone

Then, the conditional law of future path increments is completely determined by the configuration $\eta^{\text{stat}}$ at the base of the cone (= a finite window around the current position)
Remarks

- Technique is robust (applies to many spatial population models in “high density” regime) but current result “conceptual” rather than practical

- Work in progress (so far in $d \geq 3$):
  A “joint regeneration” construction allows to analyse samples of size 2 (or even more) on large space-time scales (as for the contact process) and to derive an a.s. version of the CLT

- Meta-theorem (again): “Everything”$^3$ that is true for the neutral multi-type voter model is also true for the neutral multi-type spatial logistic model.

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More details can be found in


M. B., R. Sun, Low-dimensional lonely branching random walks die out, arXiv:1708.06377
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Thank you for your attention!